Referee report on the article by A.R. Osborne "Classification of homoclinic rogue waves solutions of the nonlinear Schrödingier equation"

This article is certainly relevant to the field of your journal. Its main objective is to promote for the earth science community an idea that the different most popular rogue waves-like elementary solutions of the focusing NLS equation including Kuznetsov-Ma breather or Akhmediev breathers as well as some of theirs generalisations, including dn-wave elliptic solutions can be obtained in a universal way as appropriate limits (or reductions) of the generic finite-gap theta functional solutions of the focusing NLS (FNLS) equation.

This is of course very important to explain to the earth-science community.

The author mainly restricts his discussion by the limits of the genus 2 solutions of the FNLS equation for making the presentation as simple as possible.

Therefore, I think some further comments, concerning the solutions describing n-phase modulations of the plane wave solution of the FNLS equations obtained as a limits of the higher genus theta-functional solutions, is important for potential readership.

The finite-gap solutions were first found in 1976 by Its and Kotlyarov for the purely periodic boundary conditions . This implies the strong restrictions on the branch points of the spectral curve. Removing this restrictions but keeping the same structure of the formulas one gets the solutions which are still smooth, and not necessarily periodic but rather multi-periodic functions of x and t with a finite group of periods.

See for instance [3] , pp.101-143 for the details. These solutions are described by the formulas involving the multi-dimensional Riemann theta functions The pairwise contractions of the branch points , of the related spectral curve except one pair of them leads to the trigonometric multi-periodic solutions [2, 3] .

t-peroidic Kuznetsov breather was found in 1977 using the IST method developed by Zakharov and Schabat and was rediscovered 2 years later by Ma. The x-periodic Akhmediev breather first appeared in the article by Akhmediev , Eleonski and Kulagin [1] of 1985 containing also its two phase generalisation and its rational limit: rank 2 Peregrine breather.

An appropriate limits of Kuznetsov-Ma breather or Akhmediev breather both produce the genuine Peregrine (rank 1) rational breather. In [1], the solutions were constructed by use of some special Ansatz and was expressed the doubt concerning the possibility to get similar solutions from the finite-gap theta-functional formulas.

This statement stimulated the work by Its , Rybin and Salle [2] of 1988 showing that it is indeed possible.

In [2] was found one of the possible " infinite periods " limits of generic genus $2n$ finite-gap solution , describing the general n-phase modulation of the plane wave solution of the FNLS equation. These solutions are elementary (trigonometric) multi-periodic functions of x. They tend to the plane wave solution equation of the NLS equation when $t \to \pm\infty$. These solutions depend on $3n + 1$ independent real parameters : n real periods and 2n auxiliary "phase shift" parameters . Making all the periods commensurate one still conserves $2n + 1$ independent parameters. This includes in particular as simplest possible reductions the Akhmediev breather and its two phase generalisation found in [1].

The small modification of the formulas of [2] , where $2n$ spectral parameters $z_\nu = i\lambda_\nu$ were located inside of the interval $[-i, i]$ of the complex plane, - is to set them outside of this interval of the imaginary axis. This leads to the multi-phase generalisation of of the Kuznetsov-Ma breather so that

$$\lambda_{j+n} = -\lambda_j, \lambda_j > 1,\ j = 1, ...n.$$

, which can be easily checked by the interested reader.

The related periods of the solutions with respect to t are than given by the formulas

$$T_j = \frac{2\pi}{\lambda_j\sqrt{\lambda_j^2 - 1}}$$

[1]

Rank n rational solutions might be obtained as an appropriate (very nontrivial) passage to the limit of the solutions of [2] mentioned above when $\forall j, z_j \to 1, z_{n+j} \to -1$ i.e. the "spectral part" of parameters disappears.

This opportunity was realised in a series of works by Pierre Gaillard , based on [2] , although there are many other ways to get these rational solutions and higher Peregrine breathers . See for instance [5] for further comments.

[1] In [2] the notation T_j was used for the periods with respect to x variable.

In addition, in 1990 in the work by Alfimov , Its and Kulagin [4] , (see also [3]) the multiphase modulations of the dn-wave solutions of the focusing NLS equation have been constructed and studied in some details.

This construction corresponds to a pairwise confluence of the branch points of the spectral curve except two pairs of them .

Summarising , I think that the article might be accepted by the journal modulo giving the references on the works mentioned above and containing more general results .

References

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