MODULATION INSTABILITY OF SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION

G. L. Alfimov, A. R. Its,
and N. E. Kulagin

Multiphase solutions that describe the modulation instability of spatially periodic solutions of the nonlinear Schrödinger equation are constructed.

1. Introduction

This paper continues the investigation begun in [1] of the modulation instability of the solutions of the nonlinear Schrödinger equation (NSE)

$$i\psi_t+\psi_{xx}+2|\psi|^2\psi=0. \tag{1.1}$$

A detailed study was made in [1] of single- and two-phase modulation perturbations of the simplest solution of Eq. (1.1) described by the formula

$$\psi=e^{2it+i\varphi}. \tag{1.2}$$

The corresponding generalization to the arbitrary multiphase case was obtained in [2]. In this paper we shall consider the modulation instability of the periodic solution of the NSE that comes next in complexity after (1.2), namely,

$$\psi(x,t)=\frac{1+k}{k}\mathrm{dn}\left((x-x_0)\frac{1+k}{k},\frac{2\sqrt{k}}{1+k}\right)\exp\left\{2i\frac{1+k^2}{k^2}t+i\varphi\right\}, \tag{1.3}$$

where $\mathrm{dn}(z, k)$ is the Jacobi elliptic function, a delta amplitude. To find explicit expressions describing multiphase modulation perturbation of the solution (1.3), we shall use the method of [2], which is based on the direct degeneracy of general theta-functional (finite-gap) solutions of the complexified equation (1.1). The final expressions obtained in such an approach* are very convenient for subsequent qualitative and asymptotic analysis.

2. Description of Modulation Instability of the Solution (1.3)

In our constructions, we shall take as our point of departure the well-known (see [4-6]) g-gap solutions of the system (complexified NSE)

$$iu_t+u_{xx}-2vu^2=0,\quad -iv_t+v_{xx}-2v^2u=0, \tag{2.1}$$

described by the formulas

$$\begin{aligned} u(x,t)&=A\frac{\Theta(\mathbf{V}x+\mathbf{W}t+\boldsymbol{\eta}-\mathbf{r})}{\Theta(\mathbf{V}x+\mathbf{W}t+\boldsymbol{\eta})}\exp\{iEx+iLt\},\\ v(x,t)&=\frac{4\omega_0}{A}\frac{\Theta(\mathbf{V}x+\mathbf{W}t+\boldsymbol{\eta}+\mathbf{r})}{\Theta(\mathbf{V}x+\mathbf{W}t+\boldsymbol{\eta})}\exp\{-iEx-iLt\}. \end{aligned} \tag{2.2}$$

The parameters of the solution (2.2) are the algebraic curve Γ (Riemann surface) of genus $g > 1$ determined by the equation

$$y^2=P(\lambda)\equiv\prod_{j=1}^{2g+2}(\lambda-E_j),\quad E_j\in\mathbb{C},\quad E_j\neq E_k,\quad j\neq k,$$

*There is an alternative method of investigating the degeneracies of finite-gap solutions (see, for example, [3]) based on the "degeneracy" of the scheme of finite-gap integration itself.

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 84, No. 2, pp. 163-172, August, 1990. Original article submitted November 17, 1988.

with number $A \in \mathbb{C} \setminus \{0\}$ and vector $\eta = \{\eta_\nu\}_{\nu=0}^{g-1} \in \mathbb{C}^g$. The entities that occur in the expressions (2.2) are the Riemann theta function $\Theta(p)$ $(p \in \mathbb{C}^g)$, the constant vectors $\mathbf{V} = \{V_\nu\}_{\nu=0}^{g-1}$, $\mathbf{W} = \{W_\nu\}_{\nu=0}^{g-1}$, $\mathbf{r} = \{r_\nu\}_{\nu=0}^{g-1}$, and the constants ω_0, E, and L; they are uniquely determined by Γ (by the numbers E_j) and a canonical basis, chosen arbitrarily on Γ, of oriented cycles $a_0, \ldots, a_{g-1}$, $b_0, \ldots, b_{g-1}$ with matrix of intersection indices $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Namely, we define on Γ the basis of holomorphic differentials

$$\omega_\nu(\lambda) = \frac{C_\nu^1 \lambda^{g-1} + C_\nu^2 \lambda^{g-2} + \ldots + C_\nu^g}{\sqrt{P(\lambda)}} d\lambda, \quad \nu = 0, \ldots, g-1,$$

normalized by the condition $\int\limits_{a_\mu} \omega_\nu = \delta_{\mu\nu}$, and Abelian integrals Ω_2 and Ω_3 fixed by the requirements

a) $\int\limits_{a_\mu} d\Omega_j = 0, \quad j = 2, 3, \quad \mu = 0, \ldots, g-1;$

b) Ω_j have singularities only at the points $\infty^\pm$ (these are the two points on Γ corresponding to $\lambda \sim \infty$, $y \sim \pm \lambda^{g+1}$) and asymptotic behaviors

$$\Omega_2(\lambda) = \mp(2\lambda^2 + O(1)), \quad \lambda \to \infty^\pm, \quad \Omega_3(\lambda) = \ln \lambda + o(1), \quad \lambda \to \infty^+,$$

$$\Omega_3(\lambda) = -\ln \lambda + O(1), \quad \lambda \to \infty^-.$$

Then the constants ω_0 and L are determined from the expansions

$$\Omega_3(\lambda) = -\ln \lambda + \ln \omega_0 + o(1), \quad \lambda \to \infty^-, \quad \Omega_2(\lambda) = \mp(2\lambda^2 + L/2 + o(1)), \quad \lambda \to \infty^\pm,$$

the Θ function is constructed from the matrix

$$B_{\mu\nu} = \int\limits_{b_\nu} \omega_\mu \quad (B_{\mu\nu} = B_{\nu\mu})$$

in accordance with the formula

$$\Theta(p) = \sum_{m \in \mathbb{Z}^g} \exp\{\pi i (Bm, m) + 2\pi i (p, m)\}, \quad (p, m) = \sum_{\nu=0}^{g-1} p_\nu m_\nu, \quad (Bm, m) = \sum_{\mu,\nu=0}^{g-1} B_{\mu\nu} m_\mu m_\nu,$$

and the vectors $\mathbf{V}, \mathbf{W}, \mathbf{r}$ and the constant E are given by

$$V_\nu = 2iC_\nu^1, \quad W_\nu = -4i(C_\nu^1 C/2 + C_\nu^2), \quad C = \sum_{j=1}^{2g+2} E_j, \quad r_\nu = \int\limits_{\infty^-}^{\infty^+} \omega_\nu, \quad E = C - 2\sum_{\nu=0}^{g-1} \int\limits_{a_\nu} \lambda \omega_\nu.$$

We consider the special case of g-gap solutions determined by the conditions

$$g = 2N+1, \quad N \geqslant 1, \quad E_j \in \mathbb{R}, \quad 1 = E_1 < E_2 < \ldots < E_{g+1} = 1/k, \quad 0 < k < 1,$$

$$E_{g+2} = -E_1, \quad E_{g+3} = -E_2, \ldots, E_{2g+2} = -E_{g+1}.$$

We choose the basis of cycles on Γ in the manner indicated in Fig. 1 (the continuous straight lines are the cuts that connect the branch points E_j to each other and to infinity; in the representation of the cycles paths in the upper and lower sheets correspond to the continuous and broken curves, respectively).

We subject the parameters E_2, E_3, ..., E_g to the limiting process $E_{2j}, E_{2j+1} \to \lambda_j$, $j = 1, \ldots, N$, $1 < \lambda_1 < \lambda_2 < \ldots < \lambda_N < 1/k$. Then it is obvious that $E_{2N+2+2j}, E_{2N+3+2j} \to \lambda_{N+j} \equiv -\lambda_j$, $j = 1, \ldots, N$, and the function $\sqrt{P(\lambda)}$ is transformed into

$$\sqrt{P_0(\lambda)} = \sqrt{(\lambda^2-1)(\lambda^2-1/k^2)} \prod_{j=1}^{N} (\lambda^2 - \lambda_j^2)$$

(we shall assume that $\sqrt{(\lambda^2-1)(\lambda^2-1/k^2)} > 0$ when λ belongs to the upper edge of the cut from $1/k$ to ∞ on the upper sheet). As a result, the curve Γ degenerates into the elliptic curve Γ_0, the Riemann surface of the function $\sqrt{(\lambda^2-1)(\lambda^2-1/k^2)}$. Proceeding as in [7], in which complete degeneracy (to a curve of genus zero) was obtained, we can readily show

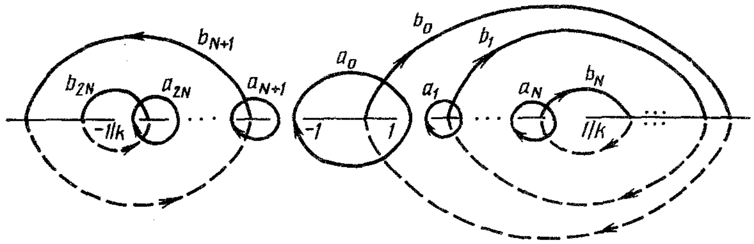

Fig. 1

that for the considered limit the infinite theta series in (2.2) are transformed into the following finite sums of one-dimensional (elliptic) theta functions:

$$\Theta(\mathbf{V}x+\mathbf{W}t+\boldsymbol{\eta}-\varepsilon\mathbf{r})\to\Theta_\varepsilon(x,t)=\sum_{\substack{m_\nu=0,1\\ \nu>0}}\Theta_3\left(-\frac{ix}{2kK}+\eta_0+\sum_j\tau_{0j}(m_j+m_{N+j})+\frac{\varepsilon}{2}\,\Big|\,\frac{\tau}{2}\right)\times$$

$$\exp\Big\{2\sum_{j>l}\tau_{jl}(m_jm_l+m_{N+j}m_{N+l})+2\sum_{j,l}\tau_{N+j,l}m_{N+j}m_l+$$

$$2i\sum_j(-i\varkappa_jx)(m_j+m_{N+j})-4\sum_j\gamma_jt(m_j-m_{N+j})+2\sum_j(\eta_j^0m_j+\eta_{N+j}^0m_{N+j})-i\varepsilon\sum_jr_j^0(m_j-m_{N+j})\Big\},$$

where

$$\Theta_3(p\,|\,\tau)=\sum_{m_0=-\infty}^{\infty}\exp\{\pi im_0^2\tau+2\pi im_0p\}$$

is the third Jacobi theta function [8],

$$\tau=\frac{iK'}{K},\quad K=\int_0^1\frac{d\lambda}{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}},\quad K'=\int_1^{1/k}\frac{d\lambda}{\sqrt{(\lambda^2-1)(1-k^2\lambda^2)}}$$

$$\tau_{jl}=k\int_{\lambda_j}^{1/k}\frac{-\varkappa_l\lambda+\beta_l}{(\lambda-\lambda_l)\sqrt{(\lambda^2-1)(1-k^2\lambda^2)}}d\lambda,\quad j>l,\quad \varkappa_j=-\frac{1}{2K}\gamma_jI_j,\quad \beta_j=-\gamma_j-\frac{1}{2K}\lambda_j\gamma_jI_j,$$

$$\tau_{N+j,l}=k\int_{\lambda_l}^{1/k}\frac{-\varkappa_j\lambda-\beta_j}{(\lambda+\lambda_j)\sqrt{(\lambda^2-1)(1-k^2\lambda^2)}}d\lambda\quad(\tau_{N+j,l}=\tau_{N+l,j}),$$

$$\gamma_j=\sqrt{(\lambda_j^2-1)(1/k^2-\lambda_j^2)},\quad I_j=\int_{-1}^{1}\frac{d\lambda}{(\lambda-\lambda_j)\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}},$$

$$r_j^0=2\int_{1/k}^{\infty}\frac{-\varkappa_j\lambda+\beta_j}{(\lambda-\lambda_j)\sqrt{(\lambda^2-1)(\lambda^2-1/k^2)}}d\lambda,\quad \tau_{0j}=\frac{i}{2K}\int_{\lambda_j}^{1/k}\frac{d\lambda}{\sqrt{(\lambda^2-1)(1-k^2\lambda^2)}}.$$

Thus, we have obtained a solution of the system (2.1) described exclusively in terms of the theory of elliptic functions:

$$u_N(x,t)=A\frac{\Theta_1(x,t)}{\Theta_0(x,t)}e^{-2i\frac{1+k^2}{k^2}t},\quad v_N(x,t)=\frac{1-k^2}{Ak^2}\frac{\Theta_{-1}(x,t)}{\Theta_0(x,t)}e^{2i\frac{1+k^2}{k^2}t}.$$

This solution is parametrized by the N real numbers λ_j, the (2N + 1) complex numbers η_0, η_j^0, η_{j+N}^0, and the complex number A, and it describes the interaction of 2N solitons of the system (2.1) on the background of its single-gap periodic solution. In particular, therefore, the functions u_N and v_N are not almost periodic with respect to the variable x. However, they do become such if the substitution $x \to ix$ is made. We show that, simultaneously, it is possible to arrange the parameters η_0, η_ν^0, and A in such a way that

$$\overline{u}_N(ix,t)=v_N(ix,t),\quad x,t\in\mathbb{R},\tag{2.3}$$

i.e., the function $\psi_N(x, t) = \bar{u}_N(ix, t)$ satisfies our NSE

$$i\psi_t+\psi_{xx}+2|\psi|^2\psi=0.$$

Noting that τ and τ_{0j} are purely imaginary and τ_{jl}, $\tau_{N+j,l}$, $\varkappa_j$, γ_j, and r_j^0 are real, we obtain

$$\bar{\Theta}_\varepsilon(ix,t)=\sum_{\substack{m_\nu=0,1\\ \nu>0}}\Theta_\varepsilon\left(\frac{x}{2kK}+\bar{\eta}_0-\sum_j\tau_{0j}(m_j+m_{N+j})+\frac{\varepsilon}{2}\middle|\frac{\tau}{2}\right)\times$$

$$\exp\Big\{2\sum_{j>l}\tau_{jl}(m_jm_l+m_{N+j}m_{N+l})+2\sum_{j,l}\tau_{N+j,l}m_{N+j}m_l-$$

$$2i\sum_j\varkappa_jx(m_j+m_{N+j})-4\sum_j\gamma_jt(m_j-m_{N+j})+2\sum_j(\bar{\eta}_j^0m_j+\bar{\eta}_{N+j}^0m_{N+j})+i\varepsilon\sum_jr_j^0(m_j-m_{N+j})\Big\}.$$

Making the substitution $m_j\to1-m_{N-j}$, $m_{N+j}\to1-m_j$ and taking into account the equation

$$\tau_{N+j,l}=\tau_{N+l,j}\,, \tag{2.4}$$

we can transform this expression for $\bar{\Theta}_\varepsilon$ to

$$\bar{\Theta}_\varepsilon(ix,t)=\exp\Big(-4i\sum_j\varkappa_jx+2\sum_{\nu>0}\eta_\nu^0+2\sum_{\nu>\mu>0}\tau_{\nu\mu}\Big)\times$$

$$\sum_{\substack{m_\nu=0,1\\ \nu>0}}\Theta_\varepsilon\left(\frac{x}{2kK}+\bar{\eta}_0+\sum_j\tau_{0j}(m_j+m_{N+j})+\varepsilon/2-2\sum_j\tau_{0j}\middle|\tau/2\right)\times$$

$$\exp\Big\{2\sum_{j>l}\tau_{jl}(m_jm_l+m_{N+j}m_{N+l})+2\sum_{j,l}\tau_{N+j,l}m_{N+j}m_l+$$

$$-2i\sum_j\varkappa_jx(m_j+m_{N+j})-4\sum_j\gamma_jt(m_j-m_{N+j})+$$

$$i\varepsilon\sum_jr_j^0(m_j-m_{N+j})-2\sum_j(\bar{\eta}_j^0m_{N+j}+\bar{\eta}_{N+j}^0m_j)-2\tau(\mathbf{m})\Big\}, \tag{2.5}$$

where (again using (2.4))

$$\tau(\mathbf{m})=\sum_{j>l}\tau_{jl}(m_j+m_l+m_{N+j}+m_{N+l})+\sum_{j,l}\tau_{N+j,l}(m_j+m_{N+l})=$$

$$\sum_{l=1}^{N-1}\sum_{j=l+1}^{N}\tau_{jl}(m_j+m_{N+j})+\sum_{l=1}^{N-1}(m_l+m_{N+l})\sum_{j=l+1}^{N}\tau_{jl}+$$

$$\sum_j(m_j+m_{N+j})\sum_l\tau_{N+j,l}=\sum_{j=2}^{N}(m_j+m_{N+j})\sum_{l=1}^{j-1}\tau_{jl}+$$

$$-\sum_{j=1}^{N-1}(m_j+m_{N+j})\sum_{l=j+1}^{N}\tau_{lj}+\sum_j(m_j+m_{N+j})\sum_l\tau_{N+j,l}=\sum_j(m_j+m_{N+j})\tau_j,$$

$$\tau_j=\sum_{l=1}^{j-1}\tau_{jl}+\sum_{l=j+1}^{N}\tau_{lj}+\sum_l\tau_{N+j,l}.$$

We impose on the parameters η and η_ν^0 the conditions

$$\operatorname{Im}\eta_0=i\sum_j\tau_{0j},\quad \operatorname{Im}\eta_j^0=\operatorname{Im}\eta_{N+j}^0,\quad \operatorname{Re}\eta_j^0+\operatorname{Re}\eta_{N+j}^0=-\tau_j.$$

Then from (2.5) we have

$$\bar{\Theta}_\varepsilon(ix,t)=\Theta_{-\varepsilon}(ix,t)\exp\Big(-4i\sum_j\varkappa_jx+2\sum_{\nu>0}\bar{\eta}_\nu^0+2\sum_{\nu>\mu>0}\tau_{\nu\mu}\Big)$$

or

$$\frac{\overline{\Theta_1(ix,t)}}{\Theta_0(ix,t)}=\frac{\Theta_{-1}(ix,t)}{\Theta_0(ix,t)}.$$

To satisfy Eq. (2.3), it is now sufficient to require $|A|=\sqrt{1-k^2}/k$.

We summarize. We have shown that for an arbitrary set of $\{\lambda_j\}$, $\{x_{0j}\}$, $\{t_{0j}\}$, x_0, and φ such that $1<\lambda_1<\lambda_2<\ldots<\lambda_N<1/k$, x_0, x_{0j}, t_{0j}, $\varphi\in\mathbb{R}$, $0<k<1$, $N\in\mathbb{N}$, we can, using the formulas

$$\psi_N(x,t)=\frac{1}{k}\sqrt{1-k^2}\,\frac{\Theta_1(x,t)}{\Theta_0(x,t)}\,e^{2i\frac{1+k^2}{k^2}t+i\varphi},$$

$$\Theta_\varepsilon(x,t)=\sum_{\substack{m_\nu=0,1\\ \nu>0}}\Theta_3\Big(\frac{x-x_0}{2kK}-\sum_j\tau_{0j}+\sum_j\tau_{0j}(m_j+m_{N+j})+$$

$$\frac{1-\varepsilon}{2}\Big|\,\tau/2\Big)\exp\Big\{2\sum_{j>l}\tau_{jl}(m_jm_l+m_{N+j}m_{N+l})+2\sum_{j,l}\tau_{N+j,l}m_{N+j}m_l+2i\sum_j\varkappa_j(x-x_{0j})(m_j+m_{N+j})-$$

$$4\sum_j\gamma_j(t-t_{0j})(m_j-m_{N+j})-\sum_j\tau_j(m_j+m_{N+j})+i\varepsilon\sum_j r_j^0(m_j-m_{N+j})\Big\},\quad \varepsilon=1,0, \tag{2.6}$$

obtain a solution of the NSE

$$i\psi_t+\psi_{xx}+2|\psi|^2\psi=0. \tag{2.7}$$

Remark 1. The parameters $\{x_{0j}\}$, $\{t_{0j}\}$, x_0 and φ are related to the original parameters η_0, η_ν^0, and A by

$$\frac{x_0}{2kK}={}^1/_2-\operatorname{Re}\eta_0,\quad x_{0j}=-\operatorname{Im}\eta_j^0/\varkappa_j,\quad \operatorname{Re}\eta_j^0=2\gamma_jt_{0j}-\tau_j/2,\quad \varphi=-\arg A.$$

The solution $\psi_N(x, t)$ is an almost-periodic function of the variable x with group of real periods $T_0=2kK$, $T_j=\pi/\varkappa_j$, $j=1,\ldots,N$. Note also that $\psi_N(x, t)$ is a smooth function of the variables x, $t\in\mathbb{R}$. This property is due to the sign of the nonlinearity in Eq. (2.7).

Now suppose that

$$\max_j\gamma_j<2\min_j\gamma_j=2\gamma_0 \tag{2.8}$$

and let us study the $t\to\pm\infty$ behavior of the solution $\psi_N(x, t)$ to accuracy $O(e^{-8\gamma_0|t|})$.

If $t\to-\infty$, then the corresponding leading terms in the sums Θ_ε correspond to vectors **m** with coordinates

$$m_1=m_2=\ldots=m_N=1,\qquad m_{N+1}=m_{N+2}=\ldots=m_{2N}=0,$$
$$m_1=0,\qquad m_2=m_3=\ldots=m_N=1,\qquad m_{N+1}=m_{N+2}=\ldots=m_{2N}=0,$$
$$m_1=m_2=\ldots=m_N=m_{N+1}=1,\qquad m_{N+2}=\ldots=m_{2N}=0,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$m_j=0,\quad m_1=\ldots=m_{j-1}=m_{j+1}=\ldots=m_N=1,\quad m_{N+1}=\ldots=m_{2N}=0,$$
$$m_1=\ldots=m_N=m_{N+j}=1,\qquad m_{N+1}=\ldots=m_{N+j-1}=m_{N+j+1}=\ldots$$
$$\ldots=m_{2N}=0,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$m_N=0,\quad m_1=\ldots=m_{N-1}=0,\quad m_{N+1}=\ldots=m_{2N}=0,$$
$$m_1=\ldots=m_N=m_{2N}=1,\qquad m_{N+1}=\ldots=m_{2N-1}=0.$$

Therefore, in the limit $t\to-\infty$ we obtain for Θ_ε

$$\Theta_\varepsilon(x,t)=\exp\Big\{2i\sum_j\varkappa_j(x-x_{0j})-4\sum_j\gamma_j(t-t_{0j})-\sum_j\tau_j+$$

$$i\varepsilon\sum_j r_j^0+2\sum_{j>l}\tau_{jl}\Big\}\Big\{\Theta_3\Big(\frac{x-x_0}{2kK}+\frac{1-\varepsilon}{2}\Big|\,\tau/2\Big)+$$

$$\sum_{j=1}^{N}\Big[\ \Theta_3\Big(\frac{x-x_0}{2kK}-\tau_{0j}+\frac{1-\varepsilon}{2}\Big|\,\tau/2\Big)\exp\{\delta_j-2i\varkappa_j(x-x_{0j})+$$

$$4\gamma_j(t-t_{0j})-i\varepsilon r_j^0\Big\}+\Theta_3\Big(\frac{x-x_0}{2kK}+\tau_{0j}+\frac{1-\varepsilon}{2}\Big|\,\tau/2\Big)\times$$

$$\exp\{\delta_j+2i\varkappa_j(x-x_{0j})+4\gamma_j(t-t_{0j})-i\varepsilon r_j^0\}\ \Big]+O(e^{8\gamma_0 t})\Big\},\qquad \delta_j=\sum_{l}\tau_{N+j,l}-\sum_{l=1}^{j-1}\tau_{jl}-\sum_{l=j+1}^{N}\tau_{lj}. \tag{2.9}$$

Similar calculations for the case $t\to+\infty$ lead to expressions that differ from (2.9) by replacement of γ_j by $-\gamma_j$ and r_j^0 by $-r_j^0$. Hence, recalling the definition of the elliptic function dn [8], we can write down the following asymptotic representation for the solution $\psi_N(x, t)$ in the limit $t\to\pm\infty$:

$$\psi_N(x,t)=\frac{1+k}{k}\,\mathrm{dn}\left((x-x_0)\frac{1+k}{k},\frac{2\sqrt{k}}{1+k}\right)e^{2i\frac{1+k^2}{k^2}t+i\varphi\pm}\times$$

$$\Big\{1+\sum_{j=1}^{N}e^{\mp4\gamma_j(t-t_{0j})}\left[a_{1j}{\pm}e^{-2i\varkappa_j(x-x_0)}+a_{2j}{\pm}e^{2i\varkappa_j(x-x_{0j})}\right]+O(e^{-8\gamma_0|t|})\Big\}, \tag{2.10}$$

where

$$\varphi^{\pm}=\varphi\mp\sum_j r_j^0,\qquad a_{\varepsilon j}{}^{\pm}\equiv a_{\varepsilon j}{}^{\pm}\left(\frac{x}{K}\right)=$$

$$e^{\delta_j}\left[\frac{\Theta_3\left(\frac{x-x_0}{2kK}+(-1)^{\varepsilon}\tau_{0j}\,|\,\tau/2\right)}{\Theta_3\left(\frac{x-x_0}{2kK}\Big|\,\tau/2\right)}e^{\pm ir_j^0}-\frac{\Theta_0\left(\frac{x-x_0}{2kK}+(-1)^{\varepsilon}\tau_{0j}\,|\,\tau/2\right)}{\Theta_0\left(\frac{x-x_0}{2kK}\Big|\,\tau/2\right)}\right],\qquad \varepsilon=1,2.$$

Thus, the constructed solution ψ_N describes an exponentially small (in the limit $|t|\to\infty$) multiphase modulation of the simplest x-periodic solution of the NSE. It is interesting that the corresponding modulation instability reduces merely to a finite change of the time phase φ:

$$\Delta\varphi=\varphi^+-\varphi^-=-2\sum_j r_j^0.$$

The phase x_0 of the spatial variable is unchanged.

<u>Remark 2.</u> The leading term, of order O(1), in the asymptotic expression (2.10) is also valid when the condition (2.8) is lifted.

<u>Remark 3.</u> For the modulus of the solution $\psi_N(x, t)$ there exists an alternative to the representation given by (2.6):

$$|\psi_N(x,t)|^2=\frac{\partial^2}{\partial x^2}\ln\Theta_0(x,t)+C,\qquad C=-\frac{2}{K}\int_0^1\frac{\lambda^2\,d\lambda}{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}}+\frac{1+k^2}{k^2}.$$

<u>Remark 4.</u> By means of the variable

$$u=\int_0^{\lambda}\frac{d\lambda}{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}},\qquad \lambda=\mathrm{sn}(u,k),$$

which uniformizes the curve Γ_0, we can go over in the solution $\psi_N(x, t)$ from the parametrization by the numbers $\{\lambda_j\}$ to parametrization by the numbers $\{u_j\}$:

$$0<u_1<u_2<\ldots<u_N<K',\qquad \lambda_j=\mathrm{sn}(iu_j+K,k).$$

The expressions for τ_{0j}, τ_{jl}, $\tau_{N+j,l}$, r_j^0, γ_j, and $\varkappa_j$ in terms of the new parameters $\{u_j\}$ are:

$$\tau_{0j}=\tau/2-(i/2K)u_j,\qquad \tau_{jl}=\frac{\pi i\tau}{2}+\frac{\pi}{2K}(u_l+u_j)-\ln\frac{\Theta_1\left(\frac{i}{4K}(u_j+u_l)\,|\,\tau/2\right)}{\Theta_1\left(\frac{i}{4K}(u_j-u_l)\,|\,\tau/2\right)},\qquad j>l,$$

$$\tau_{N+j,l}=\frac{\pi i\tau}{2}+\frac{\pi}{2K}(u_l+u_j)-\ln\frac{\Theta_2\left(\frac{i}{4K}(u_j+u_l)\,|\,\tau/2\right)}{\Theta_2\left(\frac{i}{4K}(u_j-u_l)\,|\,\tau/2\right)},$$

$$r_j^0=\operatorname{arctg}\frac{-i\,\mathrm{sn}(iu_j,k)}{1+\mathrm{dn}(iu_j,k)}\pmod{2\pi},\qquad \gamma_j=-\frac{1-k^2}{k}i\frac{\mathrm{sn}(iu_j,k)}{\mathrm{dn}^2(iu_j,k)},$$

$$\varkappa_j=-\frac{1-k^2}{k}i\frac{\mathrm{sn}(iu_j,k)}{\mathrm{cn}(iu_j,k)\,\mathrm{dn}(iu_j,k)}-\frac{i}{Kk}\Pi(K,iu_j+K+iK'),$$

where $\Pi(K, a)$ is the complete elliptic integral of the third kind [8]:

$$\Pi(u,a)=\int_0^u\frac{k^2\,\mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a\,\mathrm{sn}^2u}{1-k^2\,\mathrm{sn}^2a\,\mathrm{sn}^2u}du=$$

$$\frac{1}{2}\ln\frac{\Theta_0\left(\frac{u-a}{2K}\,\middle|\,\tau\right)}{\Theta_0\left(\frac{u+a}{2K}\,\middle|\,\tau\right)}+uZ(a),\qquad Z(u)=\frac{d}{du}\ln\Theta_0\left(\frac{u}{2K}\,\middle|\,\tau\right).$$

Remark 5. In the phase space,* the image of the solution $\psi_N(x, t)$ is a heteroclinic trajectory doubly asymptotic to two cycles — the images of the solutions $\psi(x, t)$ (1.3) differing by the phase shift $\Delta\varphi$. By the methods of this paper one can also obtain more complicated solutions that are doubly asymptotic to a torus, i.e., to a solution of the NSE that is an almost-periodic function of the variable x with two real periods. For this, it is necessary to construct in the general finite-gap expressions (2.2) a limit analogous to the one considered in this paper but corresponding to degeneracy of the curves Γ, not to the elliptic curve Γ_0, but to the hyperelliptic curve of genus 3 described by the equation

$$y^2=(\lambda^4-(\alpha^2+\alpha^{-2})\lambda^2+1)(\lambda^4-(\beta^2+\beta^{-2})\lambda^2+1). \tag{2.11}$$

As is shown in [11], the finite-gap solution of the NSE constructed from this curve can be expressed in terms of one-dimensional Jacobi theta functions and a function almost periodic with respect to x with two real periods. We intend to discuss the details of the corresponding limiting process and the expressions that describe the modulation instability of the solution corresponding to the curve (2.11) in a following publication.

LITERATURE CITED

1. N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, Zh. Eksp. Teor. Fiz., 89, 1542 (1985).
2. A. R. Its, A. V. Rybin, and M. A. Sall', Teor. Mat. Fiz., 74, 29 (1988).
3. B. A. Dubrovin, I. M. Krichever, T. G. Malanyuk, and V. G. Makhankov, "Exact solutions to a time-dependent Schrödinger equation with self-consistent potential," Preprint E5-87-710 [in English], JINR, Dubna (1987).
4. A. R. Its and V. P. Kotlyarov, Dokl. Akad. Nauk Ukr. SSR, Ser. A, No. 11, 965 (1976).
5. A. R. Its, Vestn. Leningr. Univ., No. 7, 39 (1976).
6. B. A. Dubrovin, in: Reviews of Science and Technology. Modern Problems of Mathematics, Vol. 23 [in Russian], VINITI, Moscow (1983), pp. 33–76.
7. A. R. Its, in: Problems of Mathematical Physics, No. 10 [in Russian], Leningrad State University (1983), pp. 118–137.
8. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge (1965).
9. V. M. Eleonskii, N. E. Kulagin, and N. S. Novoshilova, Teor. Mat. Fiz., 65, 391 (1985).
10. V. M. Eleonskii, N. E. Kulagin, L. M. Lerman, and Ya. L. Umanskii, Izv. Vyssh. Uchebn. Zaved. Radiofiz., 31, 149 (1988).
11. M. V. Babich, A. I. Bobenko, and V. B. Matveev, Izv. Akad. Nauk SSSR, Ser. Mat., 49, 511 (1985).

*See [9,10] for methods of introducing an infinite-dimensional phase space and the associated possibility of qualitative analysis.