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EXACT INTEGRATION OF NONLINEAR SCHRÖDINGER EQUATION

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The degeneracy of finite-gap expressions is used to obtain a many-parameter family of smooth periodic and almost periodic solutions of the nonlinear Schrödinger equation in terms of elementary functions. A scheme for obtaining these solutions by the Darboux transformation method is considered. Propagation of a soliton on an arbitrary background is studied.

Introduction

The nonlinear Schrödinger (NS) equation is one of the fundamental nonlinear equations that can be treated by the inverse scattering method [1]. This means that for this investigation one can employ the entire rich arsenal of methods of the modern theory of completely integrable systems. In particular, the problem of constructing and studying a large set of different types of exact solutions can be realized in the framework of formal schemes such as finite-gap integration (and its "degenerate" versions) and the Darboux transformation method. The stimulus to the writing of the present paper was the paper [2], which discussed some exact solutions of the NS equation. Having given an interesting physical interpretation of these solutions, the authors of [2] asserted that the obtaining of the solutions that they had found by means of the general methods of the inverse scattering technique mentioned above, in particular those like finite-gap integration, was difficult.

The present paper consists of three sections. In the first, taking degenerate general finite-gap formulas, we obtain a many-parameter family of smooth periodic and almost periodic solutions of the NS equation expressed in terms of elementary functions. These solutions are "multiphase" analogs of the solutions that appear in [2]. We also discuss the place of these solutions in the general classification of periodic solutions of the NS equation. In the second section, we consider the scheme of Darboux dressing of a periodic solution; this leads to the same results. In the third section, we dress an arbitrary solution of the NS equation and calculate the phase shift of a soliton scattered by a decreasing background and the contributions to the densities of the integrals of the motion (particle number, momentum, energy, etc.) introduced by the soliton in the case of an arbitrary background.

1. Degeneracy of Finite-Gap Formulas

All the constructions of the present section are based on the following simple remark.

ASSERTION 1.1. Suppose the functions u(x, t) and w(x, t) satisfy the system of equations

$$iu_t+u_{xx}-2wu^2=0, \quad -iw_t+w_{xx}-2uw^2=0 \tag{1.1}$$

and are extended to complex values of x to meromorphic functions. Suppose, in addition, that

$$u(ix, t)=\overline{w}(ix, t), \qquad x, t\in\mathbb{R}. \tag{1.2}$$

Then the function

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$$v(x,t)=w(ix,t), \qquad x,t\in\mathbb{R}, \tag{1.2}$$

is a smooth solution of the NS_+ equation

$$iv_t+v_{xx}+2|v|^2v=0. \tag{1.3}$$

Proof. The validity of Eq. (1.3) for the function v is trivial. The absence of real poles with respect to x for the function v(x, t) follows from the fact that substitution in (1.3) of the fraction $c/(x-x_0(t))^n$, $x, x_0\in\mathbb{R}$, $c\in\mathbb{C}$, $n\in\mathbb{Z}$, leads to the necessary equation c = 0 because of the sign in front of the nonlinearity.

Assertion 1.1 suggests a way of obtaining a rich family of solutions of Eq. (1.3) that are almost periodic with respect to x and can be expressed in terms of elementary functions. For this, it is obviously sufficient in the many-soliton (on constant background) solution of the system (1.1) to make the substitution x → ix and choose the parameters of the solution in such a way (if, of course, this is possible) that Eq. (1.2) is ensured. We now turn to the implementation of this program.

Many-soliton solutions of the system (1.1) can be obtained in a form convenient for analysis by taking the degenerate forms of its general finite-gap solutions obtained in [3,4]. The corresponding procedure is carried out in [5], and its result, given directly for the functions u(ix, t) and w(ix, t), is

$$u(ix,t)=\frac{\theta_{-1}(x,t)}{\theta_1(x,t)}\exp(-2it+i\varphi), \quad w(ix,t)=\frac{\theta_3(x,t)}{\theta_1(x,t)}\exp(2it-i\varphi),$$

$$\theta_r(x,t)=\sum_{k\in\{0,1\}^n}\exp\Big\{\sum_{\substack{\nu>\mu\\ \nu,\mu=1,\dots,n}}\ln\left[\frac{\gamma_\nu-\gamma_\mu}{\gamma_\nu+\gamma_\mu}\right]^2 k_\nu k_\mu +$$

$$\sum_{\nu=1}^{n}k_\nu\Big(i\varkappa_\nu x-2\varkappa_\nu\lambda_\nu t+(r-1)\ln\frac{\gamma_\nu-i}{\gamma_\nu+i}+\eta_\nu\Big)\Big\}, \tag{1.4}$$

$$r=-1,+1,3; \quad \gamma_\nu=\sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}, \quad \varkappa_\nu=2\sqrt{1-\lambda_\nu^2},$$

where $\eta_\nu\in\mathbb{C}$, $\varphi\in\mathbb{R}$, $\lambda_\nu\in(-1,1)$, $\lambda_\nu\neq\lambda_\mu$, $\nu=1,\dots,n\geqslant1$, are parameters of the solution. The symbol $\sum\limits_{k\in\{0,1\}^n}$ denotes summation over all n-dimensional vectors k whose coordinates k_ν are either 0 or 1. Note also that

$$u(ix,t)\,w(ix,t)=\frac{\partial^2}{\partial x^2}\ln\theta_1(x,t)$$

and, in addition, for the functions $\theta_r(x, t)$ we have determinant representations of the form

$$\theta_r(x,t)=\det(I+M_r), \quad M_{r,\nu\mu}=\frac{2\gamma_\nu}{\gamma_\nu+\gamma_\mu}\exp\Big\{(r-1)\frac{1}{2}\ln\frac{\gamma_\nu+1}{\gamma_\nu-1}\frac{\gamma_\mu+1}{\gamma_\mu-1}+$$

$$\frac{i}{2}(\varkappa_\mu+\varkappa_\nu)x-(\varkappa_\nu\lambda_\nu+\varkappa_\mu\lambda_\mu)t+\frac{1}{2}(\eta_\mu+\eta_\nu)\Big\}. \tag{1.5}$$

PROPOSITION 1.1. Under the conditions $n=2N$, $N\geqslant1$, $\lambda_{N+j}=-\lambda_j$, $0<\lambda_j<1$, $\eta_j=\eta_j^0+id_j+c_j$, $\eta_{N+j}=\eta_j^0+id_j-c_j$, $j=1,\dots,N$,

$$\eta_j^0=\sum_{\substack{l=1\\ l\neq j}}^{N}\ln\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}+\sum_{l=1}^{N}\ln\frac{\gamma_{N+j}+\gamma_l}{\gamma_{N+j}-\gamma_l}, \qquad d_j, \quad c_j\in\mathbb{R},$$

the relation (1.2) holds in formulas (1.4).

Proof. Under the considered restrictions on the parameters λ_ν and η_ν, the argument $I_r(k|x,t)$ of the exponential that occurs in the definition of the functions $\theta_r(x, t)$ can be rewritten in the form*

*In all that follows, the Latin indices take values from 1 to N, the Greek, from 1 to 2N.

$$I_r(k\,|\,x,t)=\sum_{j>l}\left(\ln\left[\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}\right]^2 k_jk_l+\ln\left[\frac{\gamma_{N+j}-\gamma_{N+l}}{\gamma_{N+j}+\gamma_{N+l}}\right]k_{N+j}k_{N+l}\right)+$$
$$\sum_{j,l}\ln\left[\frac{\gamma_{N+j}-\gamma_l}{\gamma_{N+j}+\gamma_l}\right]k_{N+j}k_l+\sum_j\,(i(\varkappa_jx+d_j)\,(k_j+k_{N+j})-$$
$$(2\varkappa_j\lambda_jt-c_j)\,(k_j-k_{N+j}))+(r-1)\sum_j\left(\ln\frac{\gamma_j-i}{\gamma_j+i}k_j+\right.$$
$$\left.\ln\frac{\gamma_{N+j}-i}{\gamma_{N+j}+i}k_{N+j}\right)+\sum_j\eta_j^0(k_j+k_{N+j}).$$

Suppose the vectors k and m are related by $m_j=1-k_{N+j}$, $m_{N+j}=1-k_j$. Then it is obvious that

$$\bar{I}_r(k(m)\,|\,x,t)=I_r(x)+\sum_{j>l}\left(\ln\left[\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}\right]^2(1-m_{N+j})\times\right.$$
$$\left.(1-m_{N+l})+\ln\left[\frac{\gamma_{N+j}-\gamma_{N+l}}{\gamma_{N+j}+\gamma_{N+l}}\right](1-m_j)\,(1-m_l)\right)+\sum_{l,j}\ln\left[\frac{\gamma_{N+j}-\gamma_l}{\gamma_{N+j}+\gamma_l}\right](1-m_j)\,(1-m_{N+l})+$$
$$\sum_j\,(i(\varkappa_jx+d_j)\,(m_j+m_{N+j})-(2\varkappa_j\lambda_jt-c_j)\,(m_j-m_{N+j}))+$$
$$(r-1)\sum_j\left(\ln\frac{\gamma_j-i}{\gamma_j+i}m_{j+N}+\ln\frac{\gamma_{N+j}-i}{\gamma_{N+j}+i}m_j\right)-\sum_j\eta_j^0(m_j+m_{N+j}),\tag{1.6}$$

where

$$I_r(x)=-2i\sum_j\,(\varkappa_jx+d_j)-(r-1)\sum_\nu\ln\frac{\gamma_\nu-i}{\gamma_\nu+i}+2\sum_j\eta_j^0.$$

We now note that under the condition $\lambda_j=-\lambda_{N+j}$

$$\frac{\gamma_j-i}{\gamma_j+i}=\frac{\gamma_{N+j}+i}{\gamma_{N+j}-i},\qquad\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}=-\frac{\gamma_{N+j}-\gamma_{N+l}}{\gamma_{N+j}+\gamma_{N+l}}\qquad\frac{\gamma_{N+j}-\gamma_l}{\gamma_{N+j}+\gamma_l}=\frac{\gamma_{N+l}-\gamma_j}{\gamma_{N+l}+\gamma_j},\tag{1.7}$$

and these equations enable us to transform (1.6) into

$$\bar{I}_r(k(m)\,|\,x,t)=I_r(x)+\sum_{\nu>\mu}\ln\left[\frac{\gamma_\nu-\gamma_\mu}{\gamma_\nu+\gamma_\mu}\right]^2+I_{2-r}(m\,|\,x,t)+I(m)-2\sum_j\eta_j^0(m_j+m_{N+j}).$$

Here

$$I(m)=-\sum_{l=1}^{N-1}\sum_{j=l+1}^{N}(m_j+m_{N+j})\ln\left[\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}\right]^2-\sum_{l=1}^{N-1}(m_l+m_{N+l})\sum_{j=l+1}^{N}\ln\left[\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}\right]^2-$$
$$\sum_{j=1}^{N}m_j\sum_{l=1}^{N}\ln\left[\frac{\gamma_{N+j}-\gamma_l}{\gamma_{N+j}+\gamma_l}\right]^2-\sum_{l=1}^{N}m_{N+l}\sum_{j=1}^{N}\ln\left[\frac{\gamma_{N+j}-\gamma_l}{\gamma_{N+j}+\gamma_l}\right]^2.$$

Taking into account (1.7) once more, we obtain

$$I(m)=-\sum_{j=2}^{N}(m_j+m_{N+j})\sum_{l=1}^{j-1}\ln\left[\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}\right]^2-\sum_{j=1}^{N-1}(m_j+m_{N+j})\sum_{l=j+1}^{N}\ln\left[\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}\right]^2-$$
$$\sum_{j=1}^{N}(m_j+m_{N+j})\sum_{l=1}^{N}\ln\left[\frac{\gamma_{N+j}-\gamma_l}{\gamma_{N+j}+\gamma_l}\right]^2=2\sum_{j=1}^{N}(m_j+m_{N+j})\left[\sum_{l=1,\ l\neq j}^{N}\ln\left[\frac{\gamma_j+\gamma_l}{\gamma_j-\gamma_l}\right]+\right.$$
$$\left.\sum_{l=1}^{N}\ln\left[\frac{\gamma_{N+j}+\gamma_l}{\gamma_{N+j}-\gamma_l}\right]\right]=2\sum_j\eta_j^0(m_j+m_{N+j}),$$

i.e.,

$$\bar{I}_r(k(m)\,|\,x,t)=I_r(x)+\sum_{\nu>\mu}\ln\left[\frac{\gamma_\nu-\gamma_\mu}{\gamma_\nu+\gamma_\mu}\right]^2+I_{2-r}(m\,|\,x,t).$$

Therefore

$$\theta_r(x,t)=\sum_{k\in\{0,1\}^{2N}}\exp I_r(k\,|\,x,t)=\sum_{m\in\{0,1\}^{2N}}\exp I_r(k(m)\,|\,x,t)=$$

$$\exp\left\{I_r(x)+\sum_{\nu>\mu}\ln\left[\frac{\gamma_\nu-\gamma_\mu}{\gamma_\nu+\gamma_\mu}\right]^2\right\}\sum_{m\in\{0,1\}^{2N}}\exp I_{2-r}(m\,|\,x,t)=E(x)\,\theta_{2-r}(x,t),$$

where by virtue of the first of Eqs. (1.7)

$$E(x)=\exp\left\{-2i\sum_j(\varkappa_jx+d_j)+2\sum_j\eta_j{}^0+\sum_{\nu>\mu}\ln\left[\frac{\gamma_\nu-\gamma_\mu}{\gamma_\nu+\gamma_\mu}\right]^2\right\}$$

and this quantity does not depend on r.

From the relation $\bar\theta_r=E\theta_{2-r}$ the identity (1.2) in Eqs. (1.4) now follows directly. We have proved Proposition 1.1.

Assertion 1.1 and Proposition 1.1 lead us to

THEOREM 1.1. Suppose $N\geqslant1$, $N\in\mathbb{Z}$. Then for any $\varphi>0$ and any set $\{\lambda_j,\,x_{0j},\,t_{0j}\}_{j=1}^N$ such that $0<\lambda_j<1$, $x_{0j},\,t_{0j}\in\mathbb{R}$, the equation $iv_t + v_{xx} + 2|v|^2v = 0$ has a solution smooth with respect to $x,\,t\in\mathbb{R}$ of the form

$$v_N(x,t)=\frac{\theta_3(x,t)}{\theta_1(x,t)}\exp(2it-i\varphi),$$

where

$$\theta_r(x,t)=\sum_{k\in\{0,1\}^{2N}}\exp\left\{\sum_{j>l}\ln\left[\frac{\gamma_j-\gamma_l}{\gamma_j+\gamma_l}\right]^2(k_jk_l+k_{N+l})+\right.$$

$$\sum_{j,l}\ln\left[\frac{1-\gamma_j\gamma_l}{1+\gamma_j\gamma_l}\right]^2k_{N+j}k_l+(r-1)\sum_j\ln\frac{\gamma_j-i}{\gamma_j+i}(k_j+k_{N+j})+$$

$$\left.\sum_j\eta_j{}^0(k_j+k_{N+1})+\sum_j[i\varkappa_j(x-x_{0j})(k_j+k_{N+j})-2\delta_j(t-t_{0j})(k_j-k_{N+j})]\right\},\quad r=1,3;\tag{1.8}$$

$$\eta_j{}^0=\sum_{l=1,\,l\neq j}^N\ln\frac{\gamma_j+\gamma_l}{\gamma_j-\gamma_l}+\sum_{l=1}^N\ln\frac{1+\gamma_j\gamma_l}{1-\gamma_j\gamma_l},\quad \gamma_j=\sqrt{\frac{1-\lambda_j}{1+\lambda_j}},\quad \varkappa_j=2\sqrt{1-\lambda_j^2},\quad \delta_j=2\lambda_j\sqrt{1-\lambda_j^2}$$

At the same time

$$|v|^2=\frac{\partial^2}{\partial x^2}\ln\theta_1(x,t).$$

In addition, if we introduce $\lambda_{N+j}=\lambda_j$, $\gamma_{N+j}=1/\gamma_j$, $\varkappa_{j+N}=\varkappa_j$, $\delta_{N+j}=-\delta_j$, $\eta^0_{N+j}=\eta_j{}^0$, $x_{0\,N+j}=x_{0j}$, $t_{0\,N+j}=t_{0j}$, then for $v_N(x,\,t)$ we can find the determinant representation

$$v_N(x,t)=\frac{\det(I+M_3)}{\det(I+M_1)}\exp(2it-i\varphi),$$

where

$$M_{r,\nu\mu}=\frac{2\gamma_\nu}{\gamma_\nu+\gamma_\mu}\exp\left\{(r-1)\frac{1}{2}\ln\frac{\gamma_\nu-i}{\gamma_\nu+i}\frac{\gamma_\mu-i}{\gamma_\mu+i}+\right.$$

$$\left.\frac{i}{2}(\varkappa_\nu+\varkappa_\mu)(x-x_{0\nu})-(\delta_\nu+\delta_\mu)(t-t_{0\nu})+\frac{1}{2}(\eta_\nu{}^0+\eta_\mu{}^0)\right\},\quad r=3,\,1.$$

Thus, we have constructed a (3N + 1)-parameter family of smooth solutions of the NS_+ equation that are almost periodic with respect to x (periods $T_j=2\pi/\varkappa_j$). These solutions are multimode (and, in the terminology of [2], multiperiodic) analogs of the simplest solutions of this type found in [2]. In our notation, the case considered in [2] corresponds to solutions $v_N(x,\,t)$ with N = 1, 2, which, as is shown in [2], describe the effects of one- and two-mode weak modulations of a stationary wave* $v_0=\exp(2it-i\varphi)$.

*In applications of the equation NS_+ to fiber optics, considered in [2], the spatial and time variables change places.

The existence of multimode generalizations of the solutions v_1 and v_2 was already noted in [2]. However, the fear was expressed there that the corresponding exact formulas would be too complicated for meaningful analysis. It appears to us that the results of the present section (obtained ultimately through the use of the general theory of finite-gap solutions of the NS_+ equation) show that the fear expressed in [2] is not substantiated. The formulas given by Theorem 1.1 for the multimode solution v_N are completely analogous to the multisoliton formulas familiar in the theory of completely integrable systems, and this, in its turn, makes it possible to investigate fairly effectively the multimode solutions $v_N(x, t)$ themselves as $t \to \pm\infty$.

THEOREM 1.2. Suppose the parameters δ_j are such that

$$\max_j \delta_j<2, \quad \min_j \delta_j=2\delta_0. \tag{1.9}$$

Then

$$v_N(x,t)=\Big[1+\sum_{l=1}^{N} A_l^{\pm} \cos \varkappa_l (x-x_{0l}) \exp(\mp 2\delta_l \mp i\alpha_l)+O(\exp(\mp 4\delta_0 t))\Big]\exp(2it-i\varphi+i\varphi^{\pm}), \tag{1.10}$$

where

$$A_l^{\pm}=-\exp(\pm 2\delta_l t_{0l})2\varkappa_l \prod_{\substack{j=1\\ j\neq l}}^{N} \frac{\gamma_l+\gamma_j}{\gamma_l-\gamma_j}\prod_{j=1}^{N}\frac{1-\gamma_l\gamma_j}{1+\gamma_l\gamma_j}, \quad \alpha_l=\operatorname{arctg}\frac{2\lambda_l}{\varkappa_l},$$

$$\varphi^{\pm}=\mp 2\sum_{j=1}^{N}\operatorname{arctg}\frac{\varkappa_j}{2\lambda_j}=\pm\sum_{j=1}^{N}\arccos\left(1-\frac{\varkappa_j^2}{2}\right).$$

The proof of Theorem 1.2 is based on the usual (see, for example, [1]) methods of asymptotic analysis of multisoliton formulas and consists of separating in the sum (1.8) the first 2N + 1 leading terms in the limit $t \to \pm\infty$. For example, in the limit $t \to -\infty$ these terms correspond to vectors k with the coordinates

$$\begin{aligned}
&1 \leftrightarrow k_1=\ldots=k_N=1, \quad k_{N+1}=\ldots=k_{2N}=0,\\
&e^{2\delta_1 t} \leftrightarrow \begin{cases} k_1=\ldots=k_N=1, \quad k_{N+1}=1, \quad k_{N+2}=\ldots=k_{2N}=0,\\ k_1=0, \quad k_2=\ldots=k_N=1, \quad k_{N+1}=\ldots=k_{2N}=0,\end{cases}\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\\
&e^{2\delta_j t} \leftrightarrow \begin{cases} k_2=\ldots=k_N=1, \; k_{N+1}=\ldots=k_{N+j-1}=k_{N+j+1}=\ldots=k_{2N}=0, \; k_{N+j}=1,\\ k_1=\ldots=k_{j-1}=k_{j+1}=\ldots=k_N=1, \; k_j=0, \; k_{N+1}=\ldots=k_{2N}=0,\end{cases}\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\\
&e^{2\delta_N t} \leftrightarrow \begin{cases} k_1=\ldots=k_N=1, \quad k_{N+1}=\ldots=k_{2N-1}=0, \quad k_{2N}=1,\\ k_1=\ldots=k_{N-1}=1, \quad k_N=0, \quad k_{N+1}=\ldots=k_{2N}=0.\end{cases}
\end{aligned} \tag{1.11}$$

To save space, we shall omit the simple algebraic manipulations that then lead ultimately to (1.10).

Remark 1.1. When the condition (1.9) on the parameters δ_j is lifted, the rough result

$$v_N(x, t)=[1+o(1)]\exp(2it+i\varphi^{\pm}), \quad t\to\pm\infty, \quad \Delta\varphi=\varphi^+-\varphi^-=-2\sum_{j=1}^{N}\arccos\left(1-\frac{\varkappa_j^2}{2}\right),$$

obviously remains valid. However, in the sum (1.8) there appear besides the terms distinguished in (1.1) additional terms of magnitude comparable to those given in (1.11). Therefore, in the general case we cannot guarantee that the solution $v_N(x, t)$ will have the simple and physically transparent (see [2]) form of the first nontrivial term of the expansion as $t \to \pm\infty$ as given by formula (1.10). In particular, such a situation will occur in the purely periodic case (multimode solution in the rigorous sense), when

$$\varkappa_j=j\varkappa,\quad \varkappa=\varkappa_0/N,\quad 0<\varkappa_0<2,\quad j=1,\dots,N. \tag{1.12}$$

It is readily understood that at large N the quantities

$$\delta_j=\frac{j\varkappa_0}{N}\sqrt{1-\frac{j^2\varkappa_0^2}{4N^2}},\quad j=1,\dots,N,$$

will not satisfy the condition (1.9). For small N, for example, for N = 2 (the case analyzed in [2]), the inequalities (1.9) can be attained. Then, in contrast to the assumption made in [2], it is more natural to expect the Fermi–Pasta–Ulam effect in the case of weak modulation of the stationary wave precisely when the periods of the perturbation are noncommensurable; assuming $\varkappa_j$ to be arbitrary points of the interval (0,2), we can readily achieve fulfillment of the inequality

$$\max_j\left[\varkappa_j\sqrt{1-\frac{\varkappa_j^2}{4}}\right]<2\min_j\left[\varkappa_j\sqrt{1-\frac{\varkappa_j^2}{4}}\right].$$

For example, for this it is sufficient to require

$$2-\sqrt{3}<\varkappa_1<\varkappa_2<\dots<\varkappa_N<\sqrt{2}.$$

We now discuss the classification of the solutions $v_N(x, t)$ in terms of the spectral properties of the corresponding Dirac operator,

$$L_N=i\sigma_3\frac{d}{dx}+\begin{pmatrix}0 & \bar v_N\\ -v_N & 0\end{pmatrix},$$

which is related to the NS_+ equation (1.3) in the framework of the inverse scattering method. In obtaining the formulas for the solution v_N, we used the procedure of going to the degenerate general finite-gap solutions as described in [5]. If we carry out the analogous procedure in the corresponding formulas (see [5]) for the Baker–Akhiezer function, we arrive at the following explicit expression for the normalized $((\psi)_1|_{x,t=0}=1)$ solution of the equation $L_N\psi=\lambda\psi$:

$$(\psi_1)=\frac{\theta_0(\lambda,x,t)}{\theta_0(\lambda,0,0)}\frac{\theta_1(0,0)}{\theta_1(x,t)}\exp[ix\sqrt{1+\lambda^2}-it2\lambda\sqrt{1+\lambda^2}\pm it],$$

$$(\psi_2)=\frac{\theta_2(\lambda,x,t)}{\theta_0(\lambda,0,0)}\frac{\theta_1(0,0)}{\theta_1(x,t)}\exp[ix\sqrt{1+\lambda^2}-it2\lambda\sqrt{1+\lambda^2+it-i\varphi}](\lambda+\sqrt{\lambda^2+1}), \tag{1.13}$$

where $\theta_1(x, t)$ is determined in (1.4), and the formulas for the functions $\theta_r(\lambda, x, t)$, $r = 0, 2$, are obtained from the corresponding formulas for $\theta_r(x, t)$ by replacing in them the parameters η_j by the functions

$$\eta_0(\lambda)=\eta_v+\ln\frac{\gamma_v+i\sqrt{1+\lambda^2}/(\lambda+i)}{\gamma_v-i\sqrt{1+\lambda^2}/(\lambda+i)}.$$

Formulas (1.13) show that for any t the spectrum of the operator L_N on $L_1(\mathbb{R})$ is continuous and is the union of the real line $\mathbb{R}$ and the interval of the imaginary axis $[-i, i]$ (Fig. 1), and that the function $\Psi(\lambda)$ can be extended to a single-valued analytic function on a Riemann surface of genus zero,

$$z^2=1+\lambda^2, \tag{1.14}$$

i.e., in accordance with the well-known (see [6]) classification of almost periodic solutions of the NS_+ equation the solutions v_N for any $N \geq 1$ are single-valued solutions. The corresponding Baker–Akhiezer functions are described by formulas (1.13). Thus, in contrast to the self-adjoint case of the NS_+ equation (see [3]), the set of single-valued solutions of the NS_+ equation contains in addition to the trivial solution $\exp[-i\varphi+2it]$ an infinite series of solutions v_N, which are labeled by the natural number N and parametrized by 3N + 1 parameters. It is obvious that a similar additional series can be constructed in the case of g-band solutions with g > 0. The reason why this additional component in the manifold of finite-gap solutions of the NS_+ equation has not hitherto been noted is curious. The point is that (see the review [7]) for smoothness and almost periodicity of a finite-gap solution it was assumed to be necessary for the degree of the divisor D of the poles of the corresponding Baker–Akhiezer function to be equal to the kind of the algebraic curve Γ along which the solution itself is constructed — a property that is

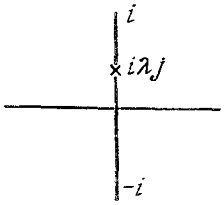

Fig. 1

clearly violated in the case of the solutions* v_N. The basis for this point of view was, first, all the accumulated experience of work with finite-gap solutions of different integrable models. This showed that if the degree of D exceeded the kind of Γ then, as a rule, one did not obtain almost periodic solutions but rather solutions that could be interpreted as soliton solutions on the background of finite-gap solutions. In addition, in the self-adjoint and purely periodic case of the NS_- equation the equality of the degree of D and the kind of Γ can be rigorously proved (see [3]). However, it was not noted [6] that the corresponding arguments (for example, such as the distribution of one of the points of the "additional" spectrum in each gap) do not extend to the nonself-adjoint case that we have with the NS_+ equation. The solutions $v_N(x, t)$ considered in the present paper show that in nonself-adjoint situations the degree of D can exceed the kind of Γ without destroying the smoothness and almost periodicity — each solution v_N is a smooth function almost periodic with respect to x with group of periods $T_j=2\pi/\varkappa_j$.

We note a further feature of the manifold of finite-gap solutions of the NS_+ equation revealed by analysis of the solutions v_N. In the purely periodic case (1.12) it follows from (1.13) that the points $\pm i\lambda_j$ are doubly degenerate eigenvalues of the periodic and antiperiodic problems for the operator L_N on the interval $(0, 2\pi/\varkappa)$. As a consequence, in the nonself-adjoint case of the NS_+ equation the manifold of g-gap solutions is determined not only by the edges of the bands of the spectrum and the divisor D but also by the degenerate gaps that occur in the finite bands of the spectrum.

All the above remarks concerning the place of the solutions v_N in general hierarchy of exact solutions of the NS_+ equation are merely of theoretical interest. With regard to applications, we are inclined, in contrast to the authors of [2], to regard the ordinary theory of finite-gap integration as rather complete; for we have obtained the "new" solutions v_N from the well-known "old" finite-gap formulas. Of course, we recognize the analytic skill of the authors of [2] and the fact that without their discovery of the solutions v_1 and v_2 we should have had no cause to reconsider the important points in the method of finite-gap integration mentioned above.

2. Use of Darboux Transformation

In [8], an analysis was made of the formulas of Darboux transformation for the system

$$iq_t=q_{xx}+2rq^2,\quad ir_t=-r_{xx}-2qr^2, \tag{2.1}$$

which differs from the system (1.1) by the substitution $q \to -q$.

It is well known [1] that the integration of the system (2.1) is based on the fact that it is the condition of simultaneous solvability of the linear system

$$\Psi_x=i\lambda\sigma_3\Psi+U\Psi, \tag{2.2}$$

$$\Psi_t=(2i\lambda^2\sigma_3+2\lambda U+V)\Psi,\quad U=\begin{pmatrix} 0 & iq \\ ir & 0 \end{pmatrix},\quad V=\begin{pmatrix} -irq & q_x \\ -r_x & irq \end{pmatrix},\quad \Psi=\begin{pmatrix} \psi \\ \varphi \end{pmatrix}. \tag{2.3}$$

Darboux transformation makes it possible to construct solutions of the system (2.1) from a known bare (r(x, t), q(x, t)). We have the following

PROPOSITION 2.1. [8]. The solution of the system (2.1) (r[N], q[N]) can be found in accordance with the formulas

*The functions $\theta_r(\lambda, x, t)$ have precisely 2N poles on the surface (1.14), and their projections onto the λ plane coincide with the points λ_ν. Therefore, the number of zeros of the function $\theta_0(\lambda, 0, 0)$ (poles of $\Psi(\lambda)$) is 2N.

$$r[N]=r+2\frac{\Delta_1(2N)}{\Delta(2N)}, \tag{2.4}$$

$$q[N]=q+2\frac{\Delta_2(2N)}{\Delta(2N)}, \tag{2.5}$$

where

$$\Delta(2N)=\det(A_{ik}),\quad A_{ik}=\begin{cases}\lambda_k^{i-1}\psi_k, & i=1,2,\ldots,N,\\ \lambda_k^{i-1-N}\varphi_k, & i=N+1,\ldots,2N,\end{cases}$$

$$\Delta_1(2N)=\det(B_{ik}),\quad B_{ik}=\begin{cases}\lambda_k^{i-1}\varphi_k, & i=1,2,\ldots,N+1,\\ \lambda_k^{i-N-2}\psi_k, & i=N+2,\ldots,2N,\end{cases}\quad k=1,2,\ldots,2N.$$

We can obtain $\Delta_2(2N)$ if in $\Delta_1(2N)$ we interchange φ_k and ψ_k; $(\psi_k, \varphi_k)^T$ is a fixed solution of the system (2.2)-(2.3) corresponding to the eigenvalue $\lambda = \lambda_k$.

The reduction of the system (2.1) and its solutions to the NS_+ equation

$$ir_t+r_{xx}+2r|r|^2=0 \tag{2.6}$$

reduces to taking

$$\lambda_{2k}=\bar{\lambda}_{2k-1},\quad \psi_{2k}=\pm\bar{\varphi}_{2k-1},\quad \varphi_{2k}=\mp\bar{\psi}_{2k-1},\quad k=1,\ 2,\ldots,N.$$

After reduction, formula (2.4) takes, when the Darboux transformation is used once (N = 1), the form

$$r[1]=r-2(\lambda-\bar{\lambda})\frac{\varphi\bar{\psi}}{|\psi|^2+|\varphi|^2}. \tag{2.7}$$

In this section we shall show how the method of Darboux dressing of the periodic solution of the NS_+ equation (2.6) yields the exact solutions found by the authors of [2]. By direct substitution one can show that Eq. (2.6) has the solution

$$r(x,t)=A\exp(is),\quad s=ax+(2A^2-a^2)t, \tag{2.8}$$

where without loss of generality $A\in\mathbb{R}$. We shall seek the solution of the system (2.2)-(2.3) (for $q = \bar{r}$) corresponding to this potential in the form

$$\Psi=\begin{pmatrix}\psi\\ \varphi\end{pmatrix}=\begin{pmatrix}e^{-is/2} & u(x,t)\\ e^{is/2} & v(x,t)\end{pmatrix}. \tag{2.9}$$

Substitution of (2.9) in (2.2) leads to a linear system with constant coefficients for the functions u(x, t) and v(x, t), the solution of which gives rise to the following cases.

A. Eigenvalues $\mu_1 = \mu_2 = 0$. The characteristic equation has the form $(\lambda + a/2)^2 = -A^2$; then

$$u=x-x_0-2at\mp\frac{1}{2A}\pm 2iAt,\quad v=\mp i\left[x-x_0-2at\pm\frac{1}{2A}\pm 2iAt\right],$$

$$r[1]=A\frac{{}^3/_4A^2-4A^2t^2-\xi^2+4it}{\xi^2+\dfrac{1}{4A^2}+4A^2t^2},\quad \xi=x-x_0-2at, \tag{2.10}$$

$$|r[1]|^2=A^2+2\frac{p(t)-\xi^2}{(p(t)+\xi^2)^2},\quad p(t)=\frac{1}{4A^2}+4A^2t^2.$$

In the limits $t \to \pm\infty$, a rational pulse propagating with velocity 2a disappears: $r[1] \to r$. We have called this solution an exulton.*

B. General Case. The characteristic equation has the form $\mu^2+(\lambda+a/2)^2+A^2=0$,

$$u=c_1\exp(\theta)+c_2\exp(-\theta),\quad \theta=\mu(x+(2\lambda-a)t),$$

$$v=c_1\frac{\mu-i(\lambda+a/2)}{iA}\exp(\theta)-c_2\frac{\mu+i(\lambda+a/2)}{iA}\exp(\theta).$$

*From the Latin exilio, to jump out or appear suddenly (the term was proposed by L. S. Mokroborodova).

For brevity, using (3.16) (see below), we write down the formula only for the square of the modulus of the solution:

$$|r[1]|^2=A^2+\frac{\partial^2}{\partial x^2}\ln\Big\{\operatorname{ch}(2\mu_1(x-at)+4(\lambda_1\mu_1-\lambda_2\mu_2)t)+$$
$$\sqrt{\frac{A^2+|\lambda+a/2|^2-|\mu|^2}{A^2+|\lambda+a/2|^2+|\mu|^2}}\cos(2\mu_2(x-at)+4(\lambda_2\mu_1+\lambda_1\mu_2)t+\varphi_0)\Big\}, \tag{2.11}$$

where $\mu=\mu_1+i\mu_2$, $\lambda=\lambda_1+i\lambda_2$. We merely note one special case of formula (2.11):

$$\lambda_1=-a/2,\quad \mu_1=0,\quad \mu_2=\sqrt{A^2-\lambda^2},\quad |A|>|\lambda_2|,$$
$$|r[1]|^2=A^2-\frac{4|\lambda_2/A|^2(A^2-\lambda_2^2)+4|\lambda_2/A|^2(A^2-\lambda_2^2)\operatorname{ch}p\cos q}{[\operatorname{ch}p+|\lambda_2/A|\cos q]^2},$$
$$p=4\lambda_2\sqrt{A^2-\lambda_2^2}t,\quad q=\sqrt{A^2-\lambda_2^2}(x-2at).$$

For fixed t, this solution is periodic with respect to x, and as $t\to\pm\infty$ we have $r[1]\to r$. Thus, this solution describes a process of self-excitation and damping of an x-periodic wave.

Many-soliton solutions on a periodic background, solutions of the type of periodic modulations, exultons, and their interaction with one another can be described by formula (2.4) (with allowance for the reduction to the NS_+ equation), where as $(\psi_k, \varphi_k)^T$ it is necessary to take the functions given by (2.9). At the same time, $(u_k, v_k)^T$ are chosen in accordance with A) and B). The scheme of Darboux dressing in the case of the NS_- equation is based on formulas (2.4) and (2.5) with imposition of the reduction

$$q=-\bar{r},\quad \lambda_{2k}=\bar{\lambda}_{2k-1},\quad \psi_{2k}=\bar{\varphi}_{2k-1},\quad \varphi_{2k}=\bar{\psi}_{2k-1},\quad k=1,2,\dots,N.$$

This enables us in an entirely similar manner to dress the zeroth or periodic solution of the NS_- equation and obtain large classes of, in general, singular solutions.

We note finally that formulas (2.4) and (2.5) also make it possible to dress finite-gap solutions. For this, it is sufficient to use the corresponding expressions for the Baker–Akhiezer function. We will obtain different forms of solutions on a finite-gap background, and the particular form of the solution (soliton, exulton, etc.) will be determined by means of the arguments of Sec. 1.

3. NS_+ Soliton on an Arbitrary Background

In this section, we extend the technique of obtaining solutions on an arbitrary background that was first employed for the Korteweg–de Vries (KdV) equations [9], and we study the question of the interaction of a soliton of the nonlinear Schrödinger equation (2.6) with an arbitrary small solution of the NS_+ equation (background). For this, we shall seek a solution of the system (2.2)-(2.3) (under the condition of reduction to the NS_+ equation: $r = \bar{q}$) corresponding to arbitrary potential $r(x, t)$ in the form

$$\psi=c_1\exp\left(\frac{\varkappa}{2}x-i\frac{\varkappa^2}{2}t+n_1\right)+c_2h_2\exp\left(-\frac{\varkappa}{2}x+\frac{i\varkappa^2}{2}t\right), \tag{3.1}$$

$$\varphi=c_1h_1\exp\left(\frac{\varkappa}{2}x-\frac{i\varkappa^2}{2}t\right)+c_2\exp\left(-\frac{\varkappa}{2}x+\frac{i\varkappa^2}{2}t+n_2\right), \tag{3.2}$$

where c_1 and c_2 are arbitrary constants. We shall seek the functions $n_1(x,t,\varkappa)$, $n_2(x,t,\varkappa)$, $h_1(x,t,\varkappa)$, $h_2(x,t,\varkappa)$ in the form of the asymptotic series

$$n_1=-\sum_{n=1}^{\infty}\frac{1}{\varkappa^n}\int_x^{\infty}f_n(x',t)\,dx',\qquad n_2=\sum_{n=1}^{\infty}\frac{1}{\varkappa^n}\int_{-\infty}^{x}g_n(x',t)\,dx', \tag{3.3}$$

$$h_1=\sum_{n=1}^{\infty}\frac{b_n}{\varkappa^n},\qquad h_2=\sum_{n=1}^{\infty}\frac{c_n}{\varkappa^n},\qquad \varkappa=2i\lambda. \tag{3.4}$$

Substitution of (3.1) and (3.2) in the linear system leads to the recursion relations

$$f_{k+1}=-\bar{r}\frac{\partial}{\partial x}\left(\frac{f_k}{\bar{r}}\right)-\sum_{j=1}^{k-1}f_jf_{k-j},\qquad k\geqslant 2,\qquad f_1=-|r|^2,\qquad f_2=\bar{r}r_x, \tag{3.5}$$

$$g_k=(-1)^k\bar{f}_k, \quad k\geqslant 1, \tag{3.6}$$

$$(b_{k+1})_x f_1+b_{k+1}f_2=-\sum_{j=1}^{k}(b_j f_{k-j+3}+(b_j)f_{k-j+2}), \tag{3.7}$$

$$c_k=(-1)^k\bar{b}_k. \tag{3.8}$$

At the same time

$$b_1=ir, \quad b_2=-ir_x+ir\int_x^\infty |r|^2\,dx'-\frac{ir}{2}\int_{-\infty}^{\infty}|r|^2\,dx.$$

The recursion relation (3.5) arose for the first time in the pioneering paper [10]; g_n and f_n are polynomial densities of the integrals of the motion.

Now, when ψ and φ corresponding to the arbitrary background r(x, t) have been constructed, we Darboux dress this background in accordance with formula (2.7). For this, we substitute (3.1) and (3.2) in (2.7), and we obtain

$$r[1]=r-4i\beta\frac{A_1\exp(i\chi)+B_1\exp(-i\chi)+A_2\exp(\theta)+B_2\exp(-\theta)}{A_3\exp(i\chi)+B_3\exp(-i\chi)+A_4\exp(\theta)+B_4\exp(-\theta)} \tag{3.9}$$

where

$$\chi=2\alpha(x-x_0)+4(\alpha^2-\beta^2)t,\ \theta=2\beta(x-x_0)+8\alpha\beta t, \quad \varkappa=2i\lambda=-2\beta+2i\alpha, \tag{3.10}$$

$$A_1=h_1\bar{h}_2=\left(\frac{ir}{\varkappa}+\frac{1}{\varkappa^2}\left(-ir_x+ir\int_x^\infty|r|^2\,dx'-\frac{ir}{2}\int_{-\infty}^{\infty}|r|^2\,dx\right)+\right.$$
$$\left.\ldots\right)\left(\frac{-i\bar{r}}{\bar{\varkappa}}+\frac{1}{\varkappa^2}\left(-i\bar{r}_x-i\bar{r}\int_{-\infty}^{x}|r|^2\,dx'+\frac{i\bar{r}}{2}\int_{-\infty}^{\infty}|r|^2\,dx\right)+\ldots\right),$$

$$B_1=\exp(\bar{n}_1+n_2)=\exp\left(\int_x^\infty\left(\frac{|r|^2}{\varkappa}-\frac{r\bar{r}_x}{\bar{\varkappa}^2}+\ldots\right)dx'+\int_{-\infty}^{x}\left(\frac{|r|^2}{\varkappa}+\frac{r\bar{r}_x}{\varkappa^2}+\ldots\right)dx'\right),$$

$$A_2=\bar{h}_2\exp n_2=\left(\frac{-ir}{\varkappa}+\frac{1}{\varkappa^2}\left(-ir_x-ir\int_{-\infty}^{x}|r|^2\,dx'+\frac{ir}{2}\int_{-\infty}^{\infty}|r|^2\,dx\right)+\ldots\right)\exp\int_{-\infty}^{x}\left(\frac{|r|^2}{\varkappa}+\frac{r\bar{r}_x}{\varkappa^2}+\ldots\right)dx,$$

$$B_2=h_1\exp\bar{n}_1=\left(\frac{ir}{\varkappa}+\frac{1}{\varkappa^2}\left(-ir_x+ir\int_x^\infty|r|^2\,dx'-\right.\right.$$
$$\left.\left.\frac{ir}{2}\int_{-\infty}^{\infty}|r|^2\,dx\right)+\ldots\right)\exp\int_x^\infty\left(\frac{|r|^2}{\bar{\varkappa}}-\frac{r\bar{r}_{x'}}{\bar{\varkappa}^2}+\ldots\right)dx',$$

$$B_3=\bar{A}_3, \quad A_3=\bar{h}_2\exp n_1+h_1\exp\bar{n}_2=\left(\frac{-i\bar{r}}{\bar{\varkappa}}+\frac{1}{\bar{\varkappa}^2}\left(-ir_x-ir\int_{-\infty}^{x}|r|^2\,dx'+\right.\right.$$
$$\left.\left.\frac{ir}{2}\int_{-\infty}^{\infty}|r|^2\,dx\right)+\ldots\right)\exp\int_x^\infty\left(\frac{|r|^2}{\varkappa}-\frac{\bar{r}r_x}{\varkappa^2}+\ldots\right)dx'+\ldots,$$

$$A_4=h_2\bar{h}_2+\exp(n_2+\bar{n}_2)=\left(\frac{i\bar{r}}{\varkappa}+\frac{1}{\varkappa^2}\left(i\bar{r}_x+i\bar{r}\int_{-\infty}^{x}|r|^2\,dx'-\right.\right.$$
$$\left.\left.\frac{i\bar{r}}{2}\int_{-\infty}^{\infty}|r|^2\,dx\right)+\ldots\right)\left(-\frac{ir}{\bar{\varkappa}}+\frac{1}{\bar{\varkappa}^2}\left(-ir_x-ir\int_{-\infty}^{x}|r|^2\,dx'+\right.\right.$$
$$\left.\left.\frac{ir}{2}\int_{-\infty}^{\infty}|r|^2\,dx\right)+\ldots\right)\exp\int_{-\infty}^{x}\left(|r|^2\frac{\varkappa+\bar{\varkappa}}{|\varkappa|^2}+\frac{r\bar{r}_{x'}}{\varkappa^2}+\frac{\bar{r}r_{x'}}{\bar{\varkappa}^2}+\ldots\right)dx',$$

$$B_4=h_1\bar{h}_1+\exp(n_1+\bar{n}_1)=\left(\frac{ir}{\varkappa}+\frac{1}{\varkappa^2}\left(-ir_x+ir\int_x^\infty|r|\,dx'-\right.\right.$$
$$\left.\left.\frac{ir}{2}\int_{-\infty}^{\infty}|r|^2\,dx\right)+\ldots\right)\left(\frac{-i\bar{r}}{\bar{\varkappa}}+\frac{1}{\varkappa^2}\left(i\bar{r}_x-i\bar{r}\int_x^\infty|r|^2\,dx'+\right.\right.$$

$$\frac{i\bar{r}}{2}\int_{-\infty}^{\infty}|r|^2dx\Big)+\ldots\Big)+\exp\int_{x}^{\infty}\Big(|r|^2\frac{\varkappa+\bar{\varkappa}}{|\varkappa|^2}-\frac{\bar{r}r_{x'}}{\varkappa^2}-\frac{r\bar{r}_{x'}}{\bar{\varkappa}^2}+\ldots\Big)dx'. \tag{3.11}$$

If we set r(x, t) = 0, then (3.9) goes over into the simplest single-soliton solution of the NS_+ equation:

$$r[1]=-2i\beta\frac{\exp(-2i(\alpha(x-x_0)+2(\alpha^2-\beta^2)t))}{\operatorname{ch}(2\beta(x-x_0)+8\alpha\beta t)}. \tag{3.12}$$

Thus, we can interpret the solution (3.9) which we have constructed as perturbation of the single-soliton NS_+ solution by an arbitrary background.

If in Eqs. (3.1) and (3.3) we set $c_1 = 0$, then we obtain the NS_+ solution

$$r[1]=r-4i\beta\frac{\bar{h}_2\exp n_2}{h_2\bar{h}_2+\exp(n_2+\bar{n}_2)}.$$

For $c_2 = 0$, we have

$$r[1]=r-4i\beta\frac{h_1\exp\bar{n}_1}{h_1\bar{h}_1+\exp(n_1+\bar{n}_1)}.$$

We now consider the soliton phase shift following scattering by an arbitrary decreasing background.

PROPOSITION 3.1. The phase shift of a soliton on passage through a rapidly decreasing background is determined by the expression

$$\Delta\Phi=\frac{\Phi_++\Phi_-}{\beta}=\frac{1}{2\beta}\sum_{n=1}^{\infty}\frac{\bar{\varkappa}^n-\varkappa^n}{|\varkappa|^{2n}}\int_{-\infty}^{\infty}(g_n-(-1)^n\bar{g}_n)dx=$$

$$-\frac{1}{\alpha^2+\beta^2}\int_{-\infty}^{\infty}|r|^2dx+\frac{i\alpha}{(\alpha^2+\beta^2)^2}\frac{1}{2}\int_{-\infty}^{\infty}(r\bar{r}_x-r_x\bar{r})dx+\frac{2\alpha^2-\beta^2}{4(\alpha^2+\beta^2)^3}\int_{-\infty}^{\infty}(|r|^4-|r_x|^2)dx, \tag{3.13}$$

i.e., it is an asymptotic series whose coefficients are integrals of the motion of the background — the particle number, momentum, energy, etc.

Suppose r(x, t) → 0 as |x| → ∞; to be specific, we take $\alpha < 0$ and $\beta < 0$. Using the notation (3.10), we see that

$$\chi=\frac{\alpha^2+\beta^2}{\alpha}x+\frac{\alpha^2+\beta^2}{2\alpha\beta}\theta.$$

We let x and t tend to +∞ in such a way that θ remains constant. Then

$$n_1h_1h_2\to 0,\quad n_2\to n_2^\infty=\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{g_n(x,t)\,dx}{\varkappa^n}.$$

Therefore, for large positive x and t formula (3.9) goes over into

$$r[1]\infty-2i\beta\frac{\exp\{-i(\chi-\operatorname{Im}n_2^\infty)\}}{\operatorname{ch}(\theta+\Phi_+)} \tag{3.14}$$

where $\Phi_+=(n_2^\infty+\bar{n}_2^\infty)/2$. Similarly, letting x and t tend to −∞ for fixed θ, we see that for large negative x and t formula (3.9) gives

$$r[1]\infty-2i\beta\frac{\exp\{-i(\chi+\operatorname{Im}n_1^\infty)\}}{\operatorname{ch}(\theta-\Phi_-)}, \tag{3.15}$$

where

$$n_1^\infty=-\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{f_n(x,t)}{\varkappa^n}dx,\quad \Phi_-=\frac{n_1^\infty+\bar{n}_1^\infty}{2}.$$

Comparing (3.14) and (3.15), we can readily obtain the expression (3.13) for the phase shift*.

*Analogous expressions for the KdV equation in terms of scattering data were first obtained by A. B. Shabat.

PROPOSITION 3.2. For polynomial densities of the first three integrals of the motion of the NS_+ solution r[1] (soliton on an arbitrary background) the following formulas hold: for the particle number density

$$|r[1]|^2=|r|^2+\frac{\partial^2}{\partial x^2}\ln\Delta; \tag{3.16}$$

for the momentum density

$$\tfrac{1}{2}(r[1]\overline{r[1]}_x-r[1]_x\overline{r[1]})=\tfrac{1}{2}(r\bar{r}_x-r_x\bar{r})+\frac{\varkappa-\bar{\varkappa}}{2}\frac{\partial^2}{\partial x^2}\ln\Delta+\frac{\partial}{\partial x}\frac{(\bar{r}a-ra)}{2\Delta}; \tag{3.17}$$

for the energy density

$$|r[1]|^4-|r[1]_x|^2=|r|^4-|r_x|^2+\frac{(\varkappa-\bar{\varkappa})^2}{4}\frac{\partial^2}{\partial x^2}\ln\Delta+$$
$$\frac{\partial}{\partial x}\left[\frac{\Delta_x\Delta_{xx}}{\Delta^2}-\frac{2}{3}\left(\frac{\Delta_x}{\Delta}\right)^3+2|r|^2\frac{\Delta_x}{\Delta}+\frac{\varkappa-\bar{\varkappa}}{2}\frac{\bar{r}a-r\bar{a}}{\Delta}\right], \tag{3.18}$$

where $\Delta=|\psi|^2+|\varphi|^2$, $a=i(\varkappa+\bar{\varkappa})\varphi\bar{\psi}$.

The proposition is proved by direct calculation by means of (2.7).

Because the higher integrals do not have a physical meaning, we shall not give the analogous expressions for their densities.

The expressions (3.16), (3.17), and (3.18) are new, because in their derivation we did not use any assumptions about the asymptotic behaviors of $\psi(x,t)$, $\varphi(x,t)$, $r(x,t)$. Thus, the terms additional to the background densities can be interpreted as the particle number density, momentum, energy, etc., that correspond to the soliton in the background-soliton interaction situation when the background behaves arbitrarily at $\pm\infty$. Thus, we obtain the possibility of describing the soliton background to the integrals of the motion also in the situations when the energy of the background is infinite, for example, for a periodic or random background. In the case of a decreasing background, we can readily show for the solution (3.9) of the nonlinear Schrödinger equation that we have constructed, using (3.1) and (3.2) and going over to integrals, that

$$\int_{-\infty}^{\infty}|r[1]|^2\,dx=\int_{-\infty}^{\infty}|r|^2\,dx+4\beta, \tag{3.19}$$

$$\frac{1}{2}\int_{-\infty}^{\infty}(r[1]\overline{r[1]}_x-r[1]_x\overline{r[1]})\,dx=\frac{1}{2}\int_{-\infty}^{\infty}(r\bar{r}_x-r_x\bar{r})\,dx+8i\alpha\beta, \tag{3.20}$$

$$\int_{-\infty}^{\infty}(|r[1]|^4-|r[1]_x|^2)\,dx=\int_{-\infty}^{\infty}(|r|^4-|r_x|^2)\,dx+16\beta\left(\frac{\beta^2}{3}-\alpha^2\right). \tag{3.21}$$

Thus, each integral of the motion of the solution r[1](x, t) can be represented as a sum of the corresponding integrals of the background and the exact single-soliton solution (3.13) of the NSE. In the case of a rapidly decreasing background formulas (3.19), (3.20), and (3.21) can be assumed to be known and to follow from the representation of the integrals of motion in terms of the scattering data. Additivity can also be proved for the higher integrals.

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NONEQUILIBRIUM THERMODYNAMICS OF A GAS OF SOLITONS OF KINK TYPE IN QUASIONE-DIMENSIONAL SYSTEMS

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A kinetic equation for solitons of kink type in quasione-dimensional systems is constructed. Calculations and a comparative analysis are made for the Φ^4 and sine–Gordon models. For both models, possible mechanisms of relaxation of the soliton gas to equilibrium are pointed out, and the temperature dependence of the transport coefficients is found.

1. Introduction

The physics of one-dimensional objects has a number of qualitative features, and therefore crystals with quasione-dimensional nature of the ordering (magnetic, ferroelectric, etc.) attract much interest [1,2,3]. One of the most interesting differences of quasione-dimensional systems is that even to describe the low-temperature thermodynamics it is necessary to take into account not only quasilinear excitations (when we refer to magnetic systems, we shall call them magnons) but also essentially nonlinear excitations, i.e., solitons of kink type (domain boundaries) [1,2,4]. Although the density of kinks at low temperatures is small compared with the magnon density (the energy of these last is much lower), the contribution of the kinks is in a number of cases decisive — their presence, strictly speaking, destroys the long-range order in the system [5], the kinks determine the central peak of the two-time correlation functions [1,2], etc.

Soliton solutions and the part they play in the construction of the equilibrium thermodynamics of quasione-dimensional systems have been considered by a number of authors in different ways [1,4,6,7]. This resulted in the formulation of a simple and adequate phenomenological approach (for the literature, see [4]). According to this approach, the thermodynamic properties of a large class of systems can be described on the basis of the following model. In the system, there exist magnon and kink gases that are nearly ideal, and the densities of the excitations of the two types are determined by Bose and Maxwell distributions, respectively. In such a treatment, the particular properties of kinks as nonlinear extended objects are not important. The properties of more complicated solitons of bion type have also been considered in the framework of this approach [8].

In the further development of this direction, one is led naturally to consider the nonequilibrium thermodynamics of quasione-dimensional systems. Essentially, we must here analyze the establishment of equilibrium in the system consisting of kinks and magnons; in the first place, it is necessary to calculate the transport coefficients of the gas of kinks.

The classical analog of this problem is the problem of the relaxation of a small admixture of a heavy gas in a light gas in the case when collisions between the heavy particles play no role [9]. To describe the kinetics of such a system, it is sufficient to consider the collisions of the heavy particles with the light ones. Provided the momentum of the heavy particle changes little, the distribution function of the heavy particles can be described by a kinetic equation with collision integral of Fokker–Planck type. As was shown in [10,11], such a situation also obtains for a kink gas, namely, the kink distribution function f(p, x, t) can be described by the Fokker–Planck equation

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